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The cohomogeneity one Einstein equations and Painlevé analysis

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Abstract

We apply techniques of Painlevé–Kowalewski analysis to a Hamiltonian system arising from symmetry reduction of the Ricci-flat Einstein equations. In the case of doubly warped product metrics on a product of two Einstein manifolds over an interval, we show that the cases when the total dimension is 10 or 11 are singled out by our analysis. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we continue our study of reductions of the Einstein equations to ordinary differential equations [5,6]. One way to perform this reduction is to assume that a Lie group G acts isometrically on an Einstein manifold (M, g) with generic orbit type G/K of real codimension one. The Einstein equations then become a *constrained* Hamiltonian system of ODEs in a variable transverse to the orbits. The phase space of our Hamiltonian system is the cotangent bundle of the space $\mathcal{M}(G/K)$ of G-invariant Riemannian metrics on G/K equipped with its canonical symplectic structure. We stress that only those trajectories which lie on the 0-level set of the Hamiltonian are solutions of the Einstein equations.

The situation is simplest if in the isotropy representation of G/K

$$\mathfrak{g}/\mathfrak{k} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$$

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the summands are pairwise inequivalent as K-modules. To describe our Hamiltonian system more precisely in this case, we will choose a homogeneous background metric on the principal orbit. Then any other homogeneous metric on this orbit is obtained by scaling the background metric on $\mathfrak{p}_i$ by scalars f_i^2 for each i. The cohomogeneity one metric on M is described by a one-parameter family of such metrics; we now view the f_i^2 as functions of the arclength coordinate t on a geodesic orthogonal to the orbits. Now t is the time parameter for the Hamiltonian flow.

Next we define position variables $q = (q_1, \dots, q_r)$ by $e^{q_i} = f_i^2$ and let $p = (p_1, \dots, p_r)$ be the conjugate momentum variables. We obtain a symplectic isomorphism $T^*(\mathcal{M}(G/K)) \approx T^*\mathbb{R}^r$, the latter being equipped with the standard symplectic structure.

To describe the Hamiltonian, let $d = (d_1, \dots, d_r)$ where d_i is the real dimension of $\mathfrak{p}_i$, and let $n = \sum_{i=1}^r d_i$ be the dimension of the principal orbit. So $n + 1$ is the dimension of our Einstein manifold M, whose Einstein constant will be denoted by Λ. Finally, we write the scalar curvature S of the principal orbit as

$$\sum_{w \in \mathcal{W}} A_w \, e^{w \cdot q}$$

for certain nonzero constants A_w, where the index set $\mathcal{W}$ is a finite set of vectors in $\mathbb{R}^r$ (with integer components) determined by G/K.

In [5] we introduced the Hamiltonian

$$\mathsf{H} = e^{-(1/2)d \cdot q} p J p^{\mathrm{T}} + e^{(1/2)d \cdot q} \left((n-1)\Lambda - \sum_w A_w \, e^{w \cdot q} \right), \tag{1.1}$$

where the symmetric matrix J has components

$$J_{ii} = \frac{1}{n-1} - \frac{1}{d_i}, \qquad J_{ij} = \frac{1}{n-1}, \quad i \neq j.$$

Note that J defines a nondegenerate quadratic form on $\mathbb{R}^{*r}$ of signature $r - 2$.

We showed in [5, Section 1], that the Einstein equations are then just the Hamiltonian flow subject to the constraint $\mathsf{H} = 0$. The components of the equations tangent to the orbit are Hamilton's equations for H, while the constraint comes from the component of the equations normal to the orbit. The equations in mixed directions hold automatically because we are assuming the summands are pairwise distinct.

We were particularly interested in investigating the existence of conserved quantities for this constrained flow. More precisely, we sought solutions F, ϕ of the equation

$$\{F, \mathsf{H}\} = \phi \mathsf{H}. \tag{1.2}$$

When the number r of irreducible summands is greater than two, we showed that for a wide class of orbit types there are no nontrivial solutions which are polynomial in the quantities p_i, e^{q_j}. When $r = 2$, the quadratic form J splits into linear factors and our techniques for showing nonexistence did not apply. In fact there are some examples known of conserved quantities of the constrained flow in this case. Moreover, even though the

Poisson bracket in (1.2) vanishes only on $\{\mathsf{H} = 0\}$, it was possible to use these conserved quantities to integrate the Einstein equations by quadratures. These examples are as follows:

(I) $\mathcal{W} = \{(0, -1), (-1, 0)\}$. In this case G/K is a product of two (nonflat) isotropy irreducible spaces, or more generally, a product of two Einstein, nonRicci-flat, manifolds (Y_i, g_i) of dimension d_i $(i = 1, 2)$. In particular $d_i > 1$. The Hamiltonian flow (with the constraint) is now the Einstein system for a doubly warped product metric on M of the form

$$\mathrm{d}t^2 + f_1(t)^2 g_1 \oplus f_2(t)^2 g_2.$$

In [6] we found conserved quantities when $\Lambda = 0$ for the following values of d_1, d_2:

$$\{d_1, d_2\} = \{5, 5\}, \{3, 6\}, \{2, 8\}. \tag{1.3}$$

Note that in these cases M has dimension 10 or 11, and $\{d_1, d_2\}$ are the positive integral solutions of the condition $d_2(d_1 - 1) = 4d_1$.

(II) $\mathcal{W} = \{(0, -1), (1, -2)\}$. If d_2 is even, then, for certain values of d_1, this case can be realised geometrically by letting the total space of certain torus bundles over products of Fano manifolds play the role of G/K (see Remark 7.7).

When $d_1 = 1$, Bérard Bergery [3] and Page and Pope [8,9] found explicit Einstein metrics. We observed in [6, Section 1] that in this situation, for any Λ, there is a conserved quantity satisfying (1.2). In addition, in [6, Section 6] we found solutions to (1.2) whenever $\Lambda = 0$ and $d_1(d_2 - 9) = 4d_2$.

In [6] the above conditions on d_1 and d_2 arose from an ansatz for constructing solutions of (1.2) of a particularly simple form. It is not clear, however, whether these conditions are really necessary for integrability.

Let us now recall the celebrated work of Kowalewski [7] on integrable rotating rigid bodies in classical mechanics. The method she introduced to help in identifying the integrable cases has since been developed and modified by many mathematicians, most notably by Painlevé, and more recently by Ablowitz et al. [1,12]. This group of techniques is called Painlevé analysis and we shall apply it in this paper to study cases (I) and (II) for arbitrary values of d_1 and d_2.

A good exposition of the Painlevé test is given in [1,12]. It applies to any system of ODEs (indeed also to PDEs), not just Hamiltonian systems. The heuristic idea is that integrability should be associated with large families of solutions which are meromorphic with movable singularities. The steps are as follows:

1. Find the *leading powers* of the solution series in powers of the independent variable s, where $s = 0$ is a movable isolated singularity. For the system to pass the Painlevé test, we need these powers to be integral and at least one of the solution series to blow up.
2. Compute the *resonances*, i.e. the steps in the expansion at which a free parameter may enter. This corresponds to the noninvertibility of the linear operator in the recursion relation for the series expansion.
3. Check the *compatibility conditions* at each resonance, i.e. check that the recursion relation can be solved at each resonance.

4. Check that the solution series actually converge in a deleted neighbourhood of the singularity.

A system of equations is said to pass the Painlevé test if there is a family of meromorphic solutions depending on the maximal number of parameters. Such a situation is regarded as a strong indicator of integrability. However, weaker forms of the Painlevé test, such as the existence of expansions meromorphic in some *rational* power of the variable, or the existence of expansions depending on a large but nonmaximal number of parameters, have also proved valuable in indicating nice properties of a system even if these may fall short of full integrability [10,11].

We shall apply Painlevé analysis to the system of equations (2.3)–(2.6), which, together with the constraint, is equivalent under a change of variables to the Einstein equations for cases (I) and (II).

In case (I) we find that all resonances are rational precisely when $n = 9$ or 10, i.e. when the Einstein manifold has dimension 10 or 11 (for other values we have only two rational resonances).

In fact (see Theorem 6.1) if $n = 9$ or 10 then the equations have a three-parameter family of solutions with convergent Painlevé expansions in powers of a rational power of s. If $d_1 = d_2 = 5$, we actually have a full four-parameter family (see Remark 6.2) for the particular significance of the values of d_i given by (1.3)).

In case (II), we find that if d_2 is even or if $d_1(d_2 - 9) = 4d_2$, there is a full four-parameter family of convergent Painlevé expansions in rational powers of s (see Theorem 7.6) (see Section 7 for further results about nonmaximal families of Painlevé expansions).

In all cases the constraint corresponds to fixing the parameter corresponding to the top resonance.

Finally, we mention that our analysis applies equally well to the case of Lorentz metrics of the form $-\mathrm{d}t^2 + g_t$, where g_t is a path of homogeneous metrics on G/K as above. The only difference is that in the Hamiltonian (1.1) the constants Λ and A_w are then replaced by $-\Lambda$ and $-A_w$. None of our arguments uses the sign of these constants.

2. The equations

In order to apply the Painlevé test, we would like to make symplectic changes of variables such that the Einstein equations involve only polynomial terms.

To this end we first replace the Hamiltonian flow of (1.1) subject to the constraint $\mathsf{H} = 0$ by an equivalent flow of a slightly simpler Hamiltonian $\bar{\mathsf{H}}$ subject to the constraint $\bar{\mathsf{H}} = 0$. We begin by diagonalising the quadratic form defined by J.

Choose a matrix C such that

$$C^{-1}J(C^{-1})^{\mathrm{T}} = \mathrm{diag}(\mu_1, \dots, \mu_r).$$

Next we choose new symplectic coordinates α, β defined by

$$q = C\alpha, \qquad \beta = pC,$$

and define vectors $\bar{d}$, $\bar{w}$ by

$$\bar{d} = dC, \qquad \bar{w} = wC.$$

Then

$$\mathsf{H} = \mathrm{e}^{-(1/2)\bar{d}\cdot\alpha}\sum_{i=1}^{r}\mu_i\beta_i^2 + \mathrm{e}^{(1/2)\bar{d}\cdot\alpha}\left((n-1)\Lambda - \sum_w A_w\,\mathrm{e}^{\bar{w}\cdot\alpha}\right).$$

Note that $\mathrm{e}^{(1/2)\bar{d}\cdot\alpha}$ represents the volume of the principal orbit relative to the volume of the background metric. Hamilton's equations are now

$$\dot{\alpha}_k = 2\,\mathrm{e}^{-(1/2)\bar{d}\cdot\alpha}\mu_k\beta_k, \quad \dot{\beta}_k = \mathrm{e}^{(1/2)\bar{d}\cdot\alpha}\left(\sum_w(\bar{d}+\bar{w})_k A_w\,\mathrm{e}^{\bar{w}\cdot\alpha} - \bar{d}_k(n-1)\Lambda\right) + \frac{\bar{d}_k}{2}\mathsf{H}.$$

We can now introduce a new independent variable s by

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \mathrm{e}^{(1/2)\bar{d}\cdot\alpha}.$$

Denoting differentiation with respect to s by a prime, the equations become

$$\alpha_k' = 2\mu_k\beta_k, \tag{2.1}$$

$$\beta_k' = \mathrm{e}^{\bar{d}\cdot\alpha}\left(\sum_w(\bar{d}+\bar{w})_k A_w\,\mathrm{e}^{\bar{w}\cdot\alpha} - \bar{d}_k(n-1)\Lambda\right) + \frac{\bar{d}_k}{2}\bar{\mathsf{H}}, \tag{2.2}$$

where $\bar{\mathsf{H}} = \mathrm{e}^{(1/2)\bar{d}\cdot\alpha}\mathsf{H}$.

Now, when H (and hence $\bar{\mathsf{H}}$) is zero, Eqs. (2.1) and (2.2) are exactly Hamilton's equations for $\bar{\mathsf{H}}$. We have therefore deduced

Proposition 2.1. *The Einstein equations are equivalent to the Hamiltonian flow of* $\bar{\mathsf{H}}$ *with the constraint* $\bar{\mathsf{H}} = 0$.

Remark 2.2. The Hamiltonian $\bar{\mathsf{H}}$ is somewhat similar to that for r particles with exponential nonnearest neighbour interactions first introduced by Bogoyavlensky [4] and analysed in detail by Adler and van Moerbeke [2]. Two important differences, however, are that our kinetic energy term has Lorentz signature and that our potential energy term need not have exactly $r+1$ terms with positive coefficients. Nevertheless, the change of variables described in [2, p. 89], can be modified to bring our equations into a nice form for the Painlevé test.

We now specialise to the situation when the number of summands r is equal to two, and, furthermore, the set $\mathcal{W}$ of weight vectors in the formula for the scalar curvature of the principal orbit has exactly two members, which will henceforth be denoted by v and w.

As mentioned earlier, before we apply the Painlevé test we would like to further change coordinates so that the equations involve polynomial rather than exponential terms. Accordingly, we let

$$x_1 = \mathrm{e}^{(\bar{d}+\bar{v})\cdot\alpha}, \qquad x_2 = \mathrm{e}^{(\bar{d}+\bar{w})\cdot\alpha},$$

where $\bar{v} = vC$ and $\bar{w} = wC$, and let y_i denote the corresponding momentum coordinates. Denoting by U the matrix

$$\begin{pmatrix} d_1 + v_1 & d_2 + v_2 \\ d_1 + w_1 & d_2 + w_2 \end{pmatrix},$$

and letting $\bar{U} = UC$, we find that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \bar{U}^{-1} \begin{pmatrix} \log x_1 \\ \log x_2 \end{pmatrix}.$$

The momentum coordinates y satisfy

$$y_i = \frac{\sum_j \beta_j \bar{U}^{ji}}{x_i},$$

and so

$$\beta = (\, x_1 y_1 \quad x_2 y_2 \,)\bar{U}.$$

If we define

$$\xi = -\overline{dU}^{-1},$$

we obtain

$$\xi_1 \log x_1 + \xi_2 \log x_2 = -\bar{d}\bar{\alpha},$$

so

$$\mathrm{e}^{\bar{d}\cdot\alpha} = x_1^{-\xi_1} x_2^{-\xi_2}.$$

Writing A_1, A_2 for A_v, A_w respectively, we see that the modified Hamiltonian is now

$$\bar{H} = (\, x_1 y_1 \quad x_2 y_2 \,) E \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix} - A_1 x_1 - A_2 x_2 + (n-1)\Lambda x_1^{-\xi_1} x_2^{-\xi_2},$$

where $E = UJU^{\mathrm{T}}$. In particular, the entries of E can be easily computed using the formula just before Theorem 4.27 in [5].

Therefore, in the (x, y) coordinates Hamilton's equations become

$$\begin{aligned} x_1' &= 2E_{11}x_1^2 y_1 + 2E_{12}x_1 x_2 y_2, \\ x_2' &= 2E_{12}x_1 x_2 y_1 + 2E_{22}x_2^2 y_2, \\ y_1' &= -2E_{11}x_1 y_1^2 - 2E_{12}x_2 y_1 y_2 + A_1 + (n-1)\Lambda\xi_1 x_1^{-\xi_1 - 1} x_2^{-\xi_2}, \\ y_2' &= -2E_{12}x_1 y_1 y_2 - 2E_{22}x_2 y_2^2 + A_2 + (n-1)\Lambda\xi_2 x_1^{-\xi_1} x_2^{-\xi_2 - 1}. \end{aligned}$$

For the remainder of the paper we specialise to the Ricci-flat case. Setting $u_i = x_i y_i$, and rescaling x_i to set the constants A_i to 1, we finally obtain the equations

$$x_1' = 2x_1(E_{11}u_1 + E_{12}u_2), \tag{2.3}$$

$$x_2' = 2x_2(E_{12}u_1 + E_{22}u_2), \tag{2.4}$$

$$u_1' = x_1, \tag{2.5}$$

$$u_2' = x_2, \tag{2.6}$$

where the matrix E is given by

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{d_1-1}{d_1} & 1 \\ 1 & \dfrac{d_2-1}{d_2} \end{pmatrix} \tag{2.7}$$

in case (I) and by

$$E = \begin{pmatrix} \dfrac{d_2-1}{d_2} & \dfrac{d_2-2}{d_2} \\ \dfrac{d_2-2}{d_2} & \dfrac{d_1d_2-d_2-4d_1}{d_1d_2} \end{pmatrix} \tag{2.8}$$

in case (II). The modified Hamiltonian becomes

$$\bar{\mathsf{H}} = E_{11}u_1^2 + 2E_{12}u_1u_2 + E_{22}u_2^2 - x_1 - x_2. \tag{2.9}$$

(In the Lorentz case, some of the constants A_i may be negative. Our rescaling means that the corresponding x_i are negative.)

Remark 2.3. Note that while x_i, u_i are not symplectic coordinates for the standard symplectic structure on $T^*\mathbb{R}_+^2$, they are symplectic coordinates for the variable coefficient symplectic structure

$$\Omega = x_1^{-1}\,\mathrm{d}x_1 \wedge \mathrm{d}u_1 + x_2^{-1}\,\mathrm{d}x_2 \wedge \mathrm{d}u_2,$$

and the above equations are the canonical equations for $\bar{\mathsf{H}}$ in this symplectic structure. The same remark applies when Λ is nonzero.

We shall first study case (I) of Section 1, i.e. when $v = (-1, 0)$, $w = (0, -1)$. In this case we look for Ricci-flat metrics of the form

$$\mathrm{d}t^2 + f_1(t)^2 g_1 \oplus f_2(t)^2 g_2,$$

where g_i are Einstein metrics on spaces Y_i of real dimension $d_i > 1$. (If $d_i = 1$ for some i then $A_i = 0$ and we are in a situation considered by Bérard Bergery [3].) We leave case (II) to Section 7.

Remark 2.4. In terms of these new variables, the conserved quantities found in [6] are

$$F = (-E_{11}(u_1 + \tfrac{1}{2}u_2)^2 + x_1)x_1^{K_1}x_2^{K_2},$$

where

$$K_1 = -\frac{(d_1+1)(d_2-1)}{2(n-1)}, \qquad K_2 = \frac{(d_1-1)(d_2-2)}{2(n-1)},$$

and $(d_1, d_2) = (3, 6)$, $(2, 8)$ or $(5, 5)$.

We can check this directly from (2.3)–(2.6). We have chosen K_1, K_2 so that

$$-K_1 E_{11} - K_2 E_{12} = \tfrac{1}{2} E_{11}, \tag{2.10}$$

$$-K_1 E_{12} - K_2 E_{22} = E_{22}. \tag{2.11}$$

Moreover, for these values of d_i we have the relation

$$\tfrac{1}{2} E_{11} = 2(1 - E_{22}). \tag{2.12}$$

Now differentiating F, and using (2.3)–(2.6) and (2.10)–(2.12), we find after some calculation that

$$F' = E_{11}(u_1 + \tfrac{1}{2}u_2) x_1^{K_1} x_2^{K_2} \bar{\mathsf{H}}.$$

So F is a conserved quantity for the flow on the hypersurface $\bar{\mathsf{H}} = 0$.

3. Leading powers

We shall now begin the Painlevé analysis of equations (2.3)–(2.6) in case (I). Let us first find the possible leading powers for expansions for x_i, u_i about a singularity at $s = 0$. We put

$$x_1 = a_0 s^{M_1} + \cdots, \qquad x_2 = b_0 s^{M_2} + \cdots,$$

$$u_1 = c_0 s^{N_1} + \cdots, \qquad u_2 = e_0 s^{N_2} + \cdots.$$

Substituting these into (2.3)–(2.6), we find the potential dominant terms on the left- and right-hand sides of the equations are, respectively,

$$a_0 M_1 s^{M_1-1}: \qquad 2E_{11} a_0 c_0 s^{M_1+N_1}, \quad 2E_{12} a_0 e_0 s^{M_1+N_2}, \tag{3.1}$$

$$b_0 M_2 s^{M_2-1}: \qquad 2E_{12} b_0 c_0 s^{M_2+N_1}, \quad 2E_{22} b_0 e_0 s^{M_2+N_2}, \tag{3.2}$$

$$c_0 N_1 s^{N_1-1}: \qquad a_0 s^{M_1}, \tag{3.3}$$

$$e_0 N_2 s^{N_2-1}: \qquad b_0 s^{M_2}, \tag{3.4}$$

provided that $M_i, N_i \neq 0$. If some of the powers M_i or N_i are zero, the right-hand side terms must be modified.

Lemma 3.1. *None of the leading powers is zero.*

Proof. We distinguish two cases.

Case 1 ($N_1 \neq N_2$). Without loss of generality we take $N_1 < N_2$.

1. We first claim that the leading power of u_1 is nonzero.

   Assume for a contradiction that u_1 has zero leading power, i.e. $N_1 = 0$. Now if x_j does not have zero leading power, then (3.1) or (3.2) implies $M_j - 1 = M_j + N_1$, giving a contradiction. So in fact x_1, x_2 have zero leading power also. Hence u_2 must be singular, and from (2.9) we see that this contradicts the fact that the Hamiltonian is constant in s.
2. Next, we claim that either both x_1, x_2 have zero leading power or neither does.

   For if x_i has zero leading power, then let R_i denote the lowest nonzero power in its expansion. It follows that:

   $$M_i + N_1 = R_i - 1 > M_i - 1,$$

   so $N_1 > -1$.

   But if x_j does not have zero leading power, then $M_j - 1 = M_j + N_1$, and so $N_1 = -1$.
3. The remaining possibilities for the variables with zero leading power are $\{u_2, x_1, x_2\}$, $\{x_1, x_2\}$, $\{u_2\}$. In the first case u_1 must be singular so, as above, we get a contradiction by considering the Hamiltonian (2.9). In the second case (3.3) and (3.4) imply that $N_j - 1 = M_j = 0$ ($j = 1, 2$) so there is no singularity, which is a contradiction.

   In the third case (3.1)–(3.3) imply that

   $$N_1 = -1, \quad M_1 = -2, \quad -1 = E_{11}c_0, \quad M_2 = 2c_0.$$

   As $E_{11} = 1 - 1/d_1$, we see that $c_0 < -1$, so $M_2 < -2$. Letting S_2 denote the first nonzero power in the expansion of u_2, Eq. (3.4) tells us that $S_2 - 1 = M_2$, so $M_2 > -1$ and again we have a contradiction.

Case 2 ($N_1 = N_2 (= N)$).

1. If $N \neq 0$, then (3.3) and (3.4) imply $N - 1 = M_j$. So if we have a zero leading power then $N = 1$, $M_1 = M_2 = 0$ and none of the functions will have a singularity.
2. If $N = 0$ then without loss of generality $M_1 \neq 0$, so we have an unbalanced power of $M_1 - 1$ on the left-hand side of (3.1).

Proposition 3.2. *The possible leading terms in our expansion are*

$$x_1 = a_0 s^{-2} + \cdots, \qquad x_2 = b_0 s^{-2} + \cdots,$$
$$u_1 = c_0 s^{-1} + \cdots, \qquad u_2 = e_0 s^{-1} + \cdots,$$

where

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ e_0 \end{pmatrix} = \frac{1}{n-1} \begin{pmatrix} d_1 \\ d_2 \\ -d_1 \\ -d_2 \end{pmatrix}.$$

Proof.

1. As all the M_i, N_i are nonzero, we deduce from (3.3) and (3.4) that

$$M_i = N_i - 1 \quad (i = 1, 2).$$

2. If $N_1 \neq N_2$, we may assume without loss of generality that $N_1 < N_2$. Then

$$M_i - 1 = M_i + N_1,$$

so

$$N_1 = -1, \qquad M_1 = -2.$$

Equating coefficients of the dominant terms in (3.1), we see that

$$E_{11}c_0 = -1,$$

so, as $E_{11} = 1 - 1/d_1$, we deduce that $c_0 < -1$. Equating terms in (3.2) implies $M_2 = 2E_{12}c_0 = 2c_0 < -2$, and, using (1), this contradicts our assumption that $N_2 > N_1$.

3. If $N_1 = N_2 = N$, we deduce from the above (1) and (3.1) or (3.2) that

$$N = -1, \qquad M_1 = M_2 = -2.$$

(Note that $E_{11}c_0 + E_{12}e_0$ and $E_{12}c_0 + E_{22}e_0$ both cannot be zero as E_{ij} is nonsingular.)

Equating coefficients then yields

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix} \begin{pmatrix} c_0 \\ e_0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

and

$$c_0 = -a_0, \qquad e_0 = -b_0,$$

which may be solved to give the desired expressions for a_0, b_0, c_0, and e_0. □

Remark 3.3. We can also ask whether there exists a solution with a Painlevé expansion around a singularity at infinity. Taking $1/s$ as our new coordinate and studying the resulting equation around $s = 0$, we find using similar arguments to those above that it is impossible to make the leading powers balance, so no such expansion exists.

4. Resonances

We next find the resonances, i.e. the steps in the expansion for our variables at which free parameters may enter.

We put

$$x_1 = \sum_{j=0}^{\infty} a_j s^{-2+j/Q}, \qquad x_2 = \sum_{j=0}^{\infty} b_j s^{-2+j/Q},$$

$$u_1 = \sum_{j=0}^{\infty} c_j s^{-1+j/Q}, \qquad u_2 = \sum_{j=0}^{\infty} e_j s^{-1+j/Q},$$

where Q is an integer to be determined later.

Substituting the above into (2.3)–(2.6) leads to the recursion relations (for $j > 0$):

$$\begin{pmatrix} j/Q & 0 & -2E_{11}a_0 & -2E_{12}a_0 \\ 0 & j/Q & -2E_{12}b_0 & -2E_{22}b_0 \\ -1 & 0 & -1+j/Q & 0 \\ 0 & -1 & 0 & -1+j/Q \end{pmatrix} \begin{pmatrix} a_j \\ b_j \\ c_j \\ e_j \end{pmatrix}$$

$$= \begin{pmatrix} 2E_{11}\left(\sum_{i=1}^{j-1} a_i c_{j-i}\right) + 2E_{12}\left(\sum_{i=1}^{j-1} a_i e_{j-i}\right) \\ 2E_{12}\left(\sum_{i=1}^{j-1} b_i c_{j-i}\right) + 2E_{22}\left(\sum_{i=1}^{j-1} b_i e_{j-i}\right) \\ 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

We will denote the 4×4 matrix by $X(\nu)$, where $\nu = j/Q$. The resonances are exactly the values of ν for which $X(\nu)$ is singular. We calculate that

$$\det X(\nu) = \nu^4 - 2\nu^3 + \omega\nu^2 + (1-\omega)\nu - 2(1+\omega), \tag{4.2}$$

where

$$\omega = \frac{2}{n-1} - 1.$$

This quartic factorises as

$$(\nu+1)(\nu-2)\left(\nu^2 - \nu + \frac{2}{n-1}\right). \tag{4.3}$$

Note that the resonance $\nu = -1$ corresponds, as usual, to the freedom in the position of the singularity, which we have placed at $s = 0$ for convenience. Also notice that the set of resonances is preserved by the map $\nu \mapsto 1-\nu$. In fact the resonances $\nu = -1, 2$ correspond to $\nu(\nu-1) = 2$, while the other two resonances correspond to $\nu(\nu-1) = -2/(n-1)$.

The discriminant of the quadratic factor in (4.3) is $(n-9)/(n-1)$, so we have to check when the square root of this is rational; equivalently we have to check when $(n-9)(n-1)$ is a perfect square.

Lemma 4.1. *Let n be a positive integer. Then* $(n-9)(n-1)$ *is a perfect square only if* $n = 9$ *or* 10.

Proof. Suppose

$$(n-9)(n-1) = m^2 \quad (n \in \mathbb{Z}_+, m \in \mathbb{Z}_+ \cup \{0\}).$$

Let m have prime decomposition

$$m = 2^{\ell_0} p_1^{\ell_1} \cdots p_l^{\ell_l}.$$

After reordering, we have

$$n-9=2^j p_1^{2\ell_1}\cdots p_u^{2\ell_u}, \qquad n-1=2^k p_{u+1}^{2\ell_{u+1}}\cdots p_l^{2\ell_l},$$

where $j+k=2\ell_0$ and $\min(j,k)\le 3$.

If j,k are both even, then $n-9$ and $n-1$ are squares, so $n=10$. If j,k are both odd, then $\frac{1}{2}(n-9)$ and $\frac{1}{2}(n-1)$ are squares, so $n=9$. □

We summarise our conclusions as follows.

Theorem 4.2. *In all cases of* (I), *two of the resonances are* -1 *and* 2.

1. *If* $n<9$, *then the other two resonances are complex.*
2. *If* $n=9$, *they are both* $\frac{1}{2}$.
3. *If* $n=10$, *they are* $\frac{1}{3}$ *and* $\frac{2}{3}$.
4. *If* $n>10$, *they are real but irrational.*

5. Compatibility conditions

We shall now focus on the cases of (I) when the resonances are rational, i.e. $n=9,10$. The values of the resonances in these cases suggest that we seek expansions in powers of $s^{1/2}$ if $n=9$, and powers of $s^{1/3}$ if $n=10$. That is, in (4.1) we take $Q=2$ for $n=9$ and $Q=3$ for $n=10$.

We must check whether the recursion relations for these expansions can be solved at the steps corresponding to resonances. For future reference, we describe the kernel of $X(\nu)$ and its transpose when ν is a resonance.

If $\nu(\nu-1)=2$, then $\ker X(\nu)$ and $\ker X(\nu)^{\mathrm{T}}$ are spanned by

$$\begin{pmatrix}(\nu-1)d_1\\(\nu-1)d_2\\d_1\\d_2\end{pmatrix}, \qquad \begin{pmatrix}1\\1\\\nu\\\nu\end{pmatrix},$$

respectively.

If $\nu(\nu-1)=-2/(n-1)$ then $\ker X(\nu)$ and $\ker X(\nu)^{\mathrm{T}}$ are spanned by

$$\begin{pmatrix}\nu-1\\1-\nu\\1\\-1\end{pmatrix}, \qquad \begin{pmatrix}d_2\\-d_1\\\nu d_2\\-\nu d_1\end{pmatrix},$$

respectively.

Case 1 ($n=9$). In this case we have $Q=2$. The resonances $\nu=\frac{1}{2}$ and $\nu=2$ therefore correspond to the stages $j=1$ and $j=4$ in the recursion.

At $j = 1$ we just take $(a_1\, b_1\, c_1\, e_1)^{\mathrm{T}}$ to be an element of the kernel of $X(\frac{1}{2})$. For the remaining values of j except for $j = 4$, $X(\frac{1}{2}j)$ is invertible. So the only question is whether the recursion can be solved at $j = 4$.

We claim that this is always possible. In fact the following lemma only uses the condition $j = 2Q$.

Lemma 5.1. *Assuming that the compatibility conditions at all earlier steps hold, then the compatibility condition at the top resonance* $\nu = 2$ *also holds.*

Proof. We need to show that the vector on the right-hand side of (4.1) is orthogonal to the kernel of $X(2)^{\mathrm{T}}$, where $j = 2Q$. This is equivalent to the vanishing of

$$E_{11}\sum_{i=1}^{j-1} a_i c_{j-i} + E_{12}\sum_{i=1}^{j-1} a_i e_{j-i} + E_{12}\sum_{i=1}^{j-1} b_i c_{j-i} + E_{22}\sum_{i=1}^{j-1} b_i e_{j-i},$$

which, by the last two rows of (4.1), equals

$$E_{11}\sum_{i=1}^{j-1}\left(\frac{i}{Q}-1\right)c_i c_{j-i} + E_{12}\sum_{i=1}^{j-1}\left(\frac{i}{Q}-1\right)c_i e_{j-i} + E_{12}\sum_{i=1}^{j-1}\left(\frac{i}{Q}-1\right)e_i c_{j-i} + E_{22}\sum_{i=1}^{j-1}\left(\frac{i}{Q}-1\right)e_i e_{j-i}.$$

The first and fourth sums on the right-hand side above are 0. To see this, we have for the first sum

$$\sum_{i=1}^{j-1}\left(\frac{i}{Q}-1\right)c_i c_{j-i} = \sum_{i=1}^{j-1}\left(\frac{j-i}{Q}-1\right)c_{j-i} c_i = \sum_{i=1}^{j-1}\left(2-\frac{i}{Q}-1\right)c_{j-i} c_i = -\sum_{i=1}^{j-1}\left(\frac{i}{Q}-1\right)c_i c_{j-i}.$$

A similar computation applies to the fourth sum. Finally, the second and third sums cancel each other since

$$\sum_{i=1}^{j-1}\left(\frac{i}{Q}-1\right)c_i e_{j-i} = \sum_{i=1}^{j-1}\left(\frac{j-i}{Q}-1\right)c_{j-i} e_i = \sum_{i=1}^{j-1}\left(1-\frac{i}{Q}\right)c_{j-i} e_i. \qquad \square$$

So we have a three-parameter family of formal solutions; one parameter comes from the position of the singularity, the other two from the resonances at $j = 1, 4$ (i.e. at $\nu = \frac{1}{2}, 2$). (in order to obtain a family of solutions involving the full four parameters one also needs to consider solutions with logarithmic terms).

Case 2 ($n = 10$). In this case, $Q = 3$, and the positive resonances are $\nu = \frac{1}{3}, \frac{2}{3}, 2$, corresponding to the steps $j = 1, 2, 6$ in the recursion.

The recursion relation at $j = 1$ just says that $(a_1\, b_1\, c_1\, e_1)^{\mathrm{T}}$ lies in the kernel of $X(\frac{1}{3})$. So we take

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ e_1 \end{pmatrix} = \lambda \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 1 \\ -1 \end{pmatrix},$$

where λ is an arbitrary scalar.

The compatibility condition at $j = 2$ is

$$\begin{pmatrix} d_2 & -d_1 & \frac{2}{3}d_2 & -\frac{2}{3}d_1 \end{pmatrix} \begin{pmatrix} 2a_1(E_{11}c_1 + E_{12}e_1) \\ 2b_1(E_{12}c_1 + E_{22}e_1) \\ 0 \\ 0 \end{pmatrix} = 0.$$

If λ is nonzero, this simplifies to

$$d_2(E_{11} - E_{12}) + d_1(E_{12} - E_{22}) = 0,$$

which is equivalent to

$$d_1 = d_2.$$

So if $d_1 = d_2 = 5$, then we can always solve at $j = 2$. Otherwise, we can only solve if we take the parameter λ from the step $j = 1$ to be zero.

It remains to check solvability at the top resonance $j = 6$. But this again follows from Lemma 5.1.

So if $d_1 = d_2 = 5$, we have four parameters for our series solution (one from each resonance and one for the position of the singularity). For other values of d_1, d_2 with $n = 10$ we only have three parameters, from the singularity position and the resonances $\nu = \frac{2}{3}, 2$.

6. Convergence and conclusions

We shall now check that the formal series solutions obtained in the previous section actually represent genuine solutions on some open set.

We can check from our expression for $X(\nu)$ that the entries of $X(\nu)^{-1}$ are all $\mathrm{O}(1/\nu)$. In fact the entries of the matrix of cofactors are polynomials in ν of degree at most three, while the determinant of $X(\nu)$ is a quartic polynomial in ν given by (4.2). Recalling that $\nu = 2$ is the largest resonance, and that $\nu = j/Q$ where $Q = 2$ or 3, we can therefore find a constant μ such that

$$\|X(\nu)^{-1}\| \leq \frac{\mu}{j}; \quad \nu > 2. \tag{6.1}$$

We denote by $\mathbf{x}_j$ the vector (a_j, b_j, c_j, e_j). Let $\kappa = 2(E_{11} + E_{22} + 2E_{12})$. Now choose ρ such that

$$\|\mathbf{x}_i\| \leq (\mu\kappa)^{i-1}\rho^i \tag{6.2}$$

for i up to and including the last resonance.

We claim that if the estimate (6.2) holds for $i \leq j-1$, where j is greater than the last resonance, it also holds for $i = j$. For, the recursion relation (4.1) tells us that

$$\mathbf{x}_j = X(\nu)^{-1}\mathbf{v}_j,$$

where $\mathbf{v}_j$ is given by the right-hand side of (4.1). Now

$$\|\mathbf{v}_j\| \leq (4E_{12} + 2E_{11} + 2E_{22})(j-1)(\mu\kappa)^{i-1}\rho^i(\mu\kappa)^{j-i-1}\rho^{j-i} \leq (j-1)\mu^{j-2}\kappa^{j-1}\rho^j$$

so, from (6.1)

$$\|\mathbf{x}_j\| \leq \|X(\nu)^{-1}\|\|\mathbf{v}_j\| \leq (\mu\kappa)^{j-1}\rho^j,$$

proving the claim.

It now follows by induction that the estimate (6.2) holds for all i. Therefore the formal series solutions we have found are actually of the form $f(s^{1/2})$, $(f(s^{1/3})$, respectively), where f is given by a Laurent series convergent on a punctured disc about the origin.

Combining the above and the results in the previous sections, we obtain the following theorem.

Theorem 6.1. *Consider the system of equations* (2.3)–(2.6) *with E given by* (2.7).

1. *If* $n \neq 9, 10$, *there are only two rational resonances.*
2. *If* $n = 9$ *or* 10, *all resonances are rational. If* $d_1 = d_2 = 5$ *we have a family of solutions meromorphic in a rational power of s, depending on the full number of parameters* (*i.e. four*). *Otherwise, we have a three-parameter family of such solutions.*

The Einstein equations (with zero Einstein constant) for the doubly warped product metric

$$\mathrm{d}t^2 + f_1(t)^2 g_1 \oplus f_2(t)^2 g_2, \tag{6.3}$$

where (Y_i, g_i) are Einstein, nonRicci-flat, metrics, are equivalent to (2.3)–(2.6) with the added constraint that the flow should lie in the zero level set of the Hamiltonian $\bar{H}$. We shall see in Section 8 that imposing the constraint $\bar{H} = 0$ corresponds to fixing the parameter corresponding to the resonance $\nu = 2$.

So the Painlevé analysis suggests that the Einstein equations for metrics of the form (6.3) should be more tractable in dimension 10 and 11 than in other dimensions, and should be particularly well-behaved when $d_1 = d_2 = 5$.

Remark 6.2. In [5,6] we looked for conserved quantities for the Einstein equations. In particular, we looked for solutions F, ϕ to Eq. (1.2), where F, ϕ are given by expansions

$$F=\sum_{b\in\mathbb{R}^r} F_b\,\mathrm{e}^{b\cdot q},\qquad \phi=\sum_{b\in\mathbb{R}^r}\phi_b\,\mathrm{e}^{b\cdot q},$$

and F_b, ϕ_b are polynomial in p_i, and are zero for all but finitely many b. We defined the *level* of b to be $\sum b_i$.

Let us now specialise to case (I) of Section 1. As mentioned there, we were able in our earlier paper [6] to construct a conserved quantity for the three values of d_1, d_2 given in (1.3). We did this by exploiting a factorisation

$$J=(c\cdot\nabla J)\theta,$$

where c lies on the null cone of J, and θ is a certain linear form in p_i. In the expansions for the solutions F, ϕ to (1.2), c was a vector of minimal level.

Now, the factorisation of J may be written as

$$n(n-1)J=((d_2\Delta-1)p_1-(d_1\Delta+1)p_2)((d_1\Delta-1)p_2-(d_2\Delta+1)p_1),$$

where $\Delta=\sqrt{(n-1)/d_1d_2}$. A necessary condition to get conserved quantities involving *rational* powers of p_i, e^{q_i} from this strategy will therefore be that Δ is rational, or equivalently that $d_1d_2(n-1)$ is a perfect square.

If we now combine this condition with the requirement suggested by Painlevé analysis that $n+1=10$ or 11, we are left with *precisely* the three cases (1.3) studied in [6]. One of these cases is that of $d_1=d_2=5$.

7. The second family

Now we turn to case (II) of Section 1 when E is given by (2.8) in Eqs. (2.3)–(2.6). We first observe that if $d_2=2$, then $E_{12}=0$, and so the equations decouple and are trivially integrable. The case when $d_1=1$ corresponds to the Bérard Bergery–Page–Pope examples and the case $d_2=1$ is ruled out by the assumption that $A_2\neq 0$. Therefore, we will exclude these special dimensions from now on.

Performing computations similar to those in Sections 3 and 4, we now find that all leading exponents must be nonzero and that there are three possibilities for the leading terms of a Painlevé expansion as follows:

1. $N_1=N_2$.

$$\begin{pmatrix}M_1\\M_2\\N_1\\N_2\end{pmatrix}=\begin{pmatrix}-2\\-2\\-1\\-1\end{pmatrix},\qquad \begin{pmatrix}a_0\\b_0\\c_0\\e_0\end{pmatrix}=\frac{1}{n-1}\begin{pmatrix}n+d_1\\-d_1\\-n-d_1\\d_1\end{pmatrix},$$

2. $N_1 < N_2$.

$$\begin{pmatrix} M_1 \\ M_2 \\ N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2\left(\dfrac{d_2-2}{d_2-1}\right) \\ -1 \\ \dfrac{3-d_2}{d_2-1} \end{pmatrix}, \qquad \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ e_0 \end{pmatrix} = \begin{pmatrix} \dfrac{d_2}{d_2-1} \\ N_2 e_0 \\ \dfrac{d_2}{1-d_2} \\ e_0 \end{pmatrix},$$

where e_0 is a free parameter.

3. $N_2 < N_1$.

$$\begin{pmatrix} M_1 \\ M_2 \\ N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} \dfrac{2d_1(d_2-2)}{4d_1+d_2-d_1d_2} \\ -2 \\ \dfrac{d_2(d_1+1)}{4d_1+d_2-d_1d_2} \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ e_0 \end{pmatrix} = \begin{pmatrix} N_1 c_0 \\ \dfrac{-d_1d_2}{4d_1+d_2-d_1d_2} \\ c_0 \\ \dfrac{d_1d_2}{4d_1+d_2-d_1d_2} \end{pmatrix},$$

where c_0 is a free parameter and $4d_1 + d_2 - d_1d_2$ must be positive in order to arrange that $N_2 < N_1$. Note also that $N_1 = M_1 + 1$.

Case (1) is similar to that discussed in the previous sections. The recursion relation is again (4.1), but with the new values for a_0, b_0, and E_{ij}. The determinant of the coefficient matrix $X(\nu)$, where $\nu = j/Q$, factorises as

$$(\nu + 1)(\nu - 2)(\nu^2 - \nu + \eta),$$

where

$$\eta = \frac{2}{1-n}\left(1 + \frac{2d_1}{d_2}\right).$$

The upshot is that we have four rational roots if

$$(n-1)d_2((n-1)d_2 + 8(n + d_1)) \tag{7.1}$$

is a perfect square, and only two rational roots otherwise. In the former case, we have exactly two positive resonances ($\nu = 2$ and one of the roots of $\nu^2 - \nu + \eta$). The compatibility condition always holds as in Lemma 5.1, so we have a three-parameter Painlevé expansion. The proof of convergence in Section 6 carries over provided that we take absolute values of E_{ij} in the definition of κ.

Remark 7.1. We do not know how to generate in closed form all (d_1, d_2) so that (7.1) is a square. Certainly this is the case when $d_1 = 1$ or when $d_2 = 2$. One can also check that it is a square when $d_1(d_2 - 9) = 4d_2$, which is the condition under which we were able to produce generalised first integrals in [6] for this family. However, one can also check that there are other solutions of the rationality condition (7.1), e.g. $(d_1, d_2) = (23, 32), (49, 50), (56, 70), (42, 128), (49, 121)$.

In case (2), the recursion is

$$\begin{pmatrix} \nu & 0 & -2E_{11}a_0 & 0 \\ 0 & \nu & -2E_{12}b_0 & 0 \\ -1 & 0 & \nu - 1 & 0 \\ 0 & -1 & 0 & \nu + N_2 \end{pmatrix} \begin{pmatrix} a_j \\ b_j \\ c_j \\ e_j \end{pmatrix}$$

$$= \begin{pmatrix} 2E_{11}\left(\sum_{i=1}^{j-1} a_i c_{j-i}\right) + 2E_{12}\left(\sum_{i=0}^{j-Q(N_2+1)} a_i e_{j-Q(N_2+1)-i}\right) \\ 2E_{12}\left(\sum_{i=1}^{j-1} b_i c_{j-i}\right) + 2E_{22}\left(\sum_{i=0}^{j-Q(N_2+1)} b_i e_{j-Q(N_2+1)-i}\right) \\ 0 \\ 0 \end{pmatrix}. \tag{7.2}$$

The resonances are now found to be $-1, 0, -N_2, 2$ (the resonance $\nu = 0$ corresponds to the freedom in the choice of e_0). Note that $-N_2 = (d_2 - 3)/(d_2 - 1)$, so if d_2 is even it is natural to take

$$Q = d_2 - 1,$$

and hence

$$N_2 Q = 3 - d_2, \qquad Q(N_2 + 1) = 2.$$

One then checks that $\ker(X(-N_2))$ and $\ker(X(-N_2)^{\mathrm{T}})$ are spanned, respectively, by

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} (d_2 - 1)(d_2 - 3)e_0 \\ d_2(d_2 + 1) \\ (d_2 - 3)^2 e_0 \\ -N_2 d_2(d_2 + 1) \end{pmatrix},$$

and $\ker(X(2))$ and $\ker(X(2)^{\mathrm{T}})$ are spanned, respectively, by

$$\begin{pmatrix} d_2 + 1 \\ (d_2 + 1)E_{12}b_0 \\ d_2 + 1 \\ (d_2 - 1)E_{12}b_0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

Lemma 7.2. *The compatibility condition at $\nu = -N_2$ is always satisfied if d_2 is even.*

Proof. The vector on the right-hand side of the recursion at this stage (i.e. $j = -N_2Q$) is

$$2\begin{pmatrix} E_{11}\sum_{i=1}^{d_2-4} a_i c_{d_2-3-i} + E_{12}\sum_{i=0}^{d_2-5} a_i e_{d_2-5-i} \\ E_{12}\sum_{i=1}^{d_2-4} b_i c_{d_2-3-i} + E_{22}\sum_{i=0}^{d_2-5} b_i e_{d_2-5-i} \\ 0 \\ 0 \end{pmatrix}. \tag{7.3}$$

An induction using the recursion relation (7.2) shows that a_i, b_i, c_i, e_i vanish for all *odd i* less than $-N_2Q = d_2 - 3$. Since d_2 is even, it now follows that the each of the four sums in (7.3) vanishes, proving that the compatibility condition holds. □

Remark 7.3. When d_2 is odd, the compatibility condition at $\nu = -N_2$ is not always satisfied. We now take Q to be $\frac{1}{2}(d_2 - 1)$ and so $Q(N_2 + 1) = 1$ instead. Since $N_2 \neq 0$, we have $d_2 \neq 3$. When $d_2 = 5$, $-N_2Q = 1$, and the compatibility condition is given by

$$(5-1)(5-3)E_{12}a_0e_0^2 + 5(5+1)E_{22}b_0e_0 = 0.$$

Substituting the values of the constants on the left-hand side, one obtains $3(1 + 5/d_1) > 0$.

Indeed, one may use MAPLE to compute the compatibility condition at $\nu = -N_2$ for larger odd values of d_2. Writing $d_2 = 2k + 3$, the compatibility condition is, up to a factor of the form $C_k e_0^{k+1} d_1^{-k}$, where C_k is a nonzero constant, a polynomial of degree k in d_1 with integer coefficients. Hence the compatibility condition is satisfied whenever we have a root of this polynomial which is a positive integer. Checking the roots using MAPLE for $d_2 \leq 45$ we find that there are always precisely two rational roots: $-d_2$ and $4d_2/(d_2 - 9)$. Therefore the compatibility condition holds for those values of d_1 and d_2 for which we found generalised first integrals in [6, Section 5].

Lemma 7.4. *The compatibility condition at* $\nu = 2$ *holds assuming that all earlier recursion relations hold*.

Proof. We first give the proof in the case when d_2 is even.

The condition at $\nu = 2$ (i.e. $j = 2Q$) is

$$E_{11}\sum_{i=1}^{2Q-1} a_i c_{2Q-i} + E_{12}\sum_{i=0}^{2Q-2} a_i e_{2Q-2-i} = 0.$$

Now, using the relation $a_i = (i/Q - 1)c_i$ from (7.2), and an argument similar to that in Lemma 5.1, we find that the first sum is zero. The second sum becomes

$$E_{12}\sum_{i=0}^{2Q-2}\left(\frac{i}{Q} - 1\right) c_i e_{2Q-2-i}. \tag{7.4}$$

Notice that in the above step we have not used the parity of d_2.

In order to show that (7.4) is zero, we use the second equation coming from the recursion at $j = 2Q - 2$. That is

$$\left(\frac{2Q-2}{Q}\right) b_{2Q-2} - 2E_{12} b_0 c_{2Q-2} = 2E_{12} \sum_{i=1}^{2Q-3} b_i c_{2Q-2-i} + 2E_{22} \sum_{i=0}^{2Q-4} b_i e_{2Q-4-i}. \tag{7.5}$$

The rightmost sum may be rewritten as

$$\sum_{i=0}^{2Q-4} \left(N_2 + \frac{i}{Q}\right) e_i e_{2Q-4-i} = \sum_{i=0}^{2Q-4} \left(\frac{i+2}{Q} - 1\right) e_i e_{2Q-4-i}.$$

We can now change to a new index $k = 2Q - 4 - i$, and the sum becomes

$$\sum_{i=0}^{2Q-4} \left(1 - \frac{k+2}{Q}\right) e_k e_{2Q-4-k}.$$

This shows that the rightmost sum in (7.5) is zero.

Eq. (7.5) is now equivalent to

$$\left(\frac{2Q-2}{Q}\right)\left(N_2 + \frac{2Q-2}{Q}\right) e_{2Q-2} - 2E_{12} b_0 c_{2Q-2}$$
$$= 2E_{12} \sum_{i=1}^{2Q-3} \left(N_2 + \frac{i}{Q}\right) e_i c_{2Q-2-i},$$

which can be rewritten as

$$\left(\frac{2Q-2}{Q}\right) e_{2Q-2} - 2E_{12} b_0 c_{2Q-2} = 2E_{12} \sum_{i=1}^{2Q-3} \left(\frac{i+2}{Q} - 1\right) e_i c_{2Q-2-i}.$$

Next we let $k = 2Q - 2 - i$, and obtain

$$\left(\frac{2Q-2}{Q}\right) e_{2Q-2} - 2E_{12} b_0 c_{2Q-2} = 2E_{12} \sum_{k=1}^{2Q-3} \left(1 - \frac{k}{Q}\right) e_{2Q-2-k} c_k.$$

Using our expressions for b_0, c_0, e_0, and E_{12}, we see that this is

$$-2E_{12} e_{2Q-2} c_0 - 2E_{12} \left(1 - \frac{2Q-2}{Q}\right) e_0 c_{2Q-2} = 2E_{12} \sum_{k=1}^{2Q-3} \left(1 - \frac{k}{Q}\right) e_{2Q-2-k} c_k,$$

yielding the vanishing of (7.4) as required.

If d_2 is odd, then $Q = \frac{1}{2}(d_2 - 1)$. If all earlier recursion relations hold, then using again the second equation of the recursion at $j = d_2 - 2 = 2Q - 1$ in place of (7.5) in an analogous computation will give us the desired result. □

In case (3) the recursion relation is given by

$$\begin{pmatrix} \nu & 0 & 0 & -2E_{12}a_0 \\ 0 & \nu & 0 & -2E_{22}b_0 \\ -1 & 0 & \nu+N_1 & 0 \\ 0 & -1 & 0 & \nu-1 \end{pmatrix} \begin{pmatrix} a_j \\ b_j \\ c_j \\ e_j \end{pmatrix} = \begin{pmatrix} 2E_{12}(\sum_{i=1}^{j-1} a_i e_{j-i}) + 2E_{11}(\sum_{i=0}^{j-Q(N_1+1)} a_i c_{j-Q(N_1+1)-i}) \\ 2E_{22}(\sum_{i=1}^{j-1} b_i e_{j-i}) + 2E_{12}(\sum_{i=0}^{j-Q(N_1+1)} b_i c_{j-Q(N_1+1)-i}) \\ 0 \\ 0 \end{pmatrix}. \tag{7.6}$$

The resonances are $-1, 0, -N_1$ and 2. Since $-N_1$ is negative, we get at most a three-parameter family of Painlevé solutions. We will show below that we always get exactly a three-parameter family of such solutions, which are actually meromorphic if $d_2 > 3$.

First we observe that the condition $4d_1 + d_2 - d_1d_2 > 0$ can be rewritten as

$$(4 - d_2)d_1 > -d_2,$$

and if $d_2 > 4$, then we have

$$d_1 < 1 + \frac{4}{d_2 - 4}.$$

Since we are excluding the values $d_1 = 1$ and $d_2 = 1, 2$, we obtain Table 1 of possible values.

Hence $N_1 + 1 = M_1 + 2$ is a positive integer for $4 \le d_2 \le 7$ and in these cases we can take Q in the Painlevé expansions to be 1. We can take $Q = 1$ as well when $d_2 = 3 = d_1$. In these cases, note that $N_1 + 1 \ge 3$ and resonance occurs at $j = 2$. Since the right-hand side of (7.6) is 0 when $j = 1$, it follows that $a_1 = b_1 = c_1 = e_1 = 0$. Hence the right-hand side of (7.6) at $j = 2$ is also 0. So the compatibility condition holds automatically.

It remains to consider the case when $d_2 = 3$ and $d_1 > 3$. It is natural to choose $Q = d_1+3$, but in any case we need to have $2 = j_0/Q$ and $N_1 + 1 = j_1/Q$ for positive integers j_0 and j_1. Because $N_1 + 1 = 2(2d_1 + 3)/(d_1 + 3) > 2$, we have $j_0 < j_1 = Q(N_1 + 1)$. Since the

Table 1

d_2	d_1	n	$4d_1 + d_2 - d_1d_2$	M_1
3	d_1	$d_1 + 3$	$d_1 + 3$	$2d_1/(d_1 + 3)$
4	d_1	$d_1 + 4$	4	d_1
5	2	7	3	4
5	3	8	2	9
5	4	9	1	24
6	2	8	2	8
7	2	9	1	20

kernel of the transpose of the coefficient matrix in (7.6) for $\nu = 2$ is spanned by (0, 1, 0, 2), the compatibility condition is given by

$$\sum_{i=0}^{j_0-1} b_i e_{j_0-i} = 0.$$

That this holds now follows from the argument in the proof of Lemma 5.1.

Remark 7.5. The convergence proof of Section 6 will again work in cases (2) and (3), provided we let $\kappa = 2(|E_{11}| + |E_{22}| + 2|E_{12}|)$ and choose μ, ρ so that $\mu\kappa, \rho > 1$. (This is to ensure that the $(\mu\kappa)^{j-4}$, ρ^{j-2} terms in the estimate for the right-hand side of (7.2) are dominated by $(\mu\kappa)^{j-2}$, ρ^j.)

In summary we obtain the following:

Theorem 7.6. *The system of equations given by* (2.3)–(2.6), *with E given by* (2.8), *has a maximal family of solutions meromorphic in a rational power of s if one of the following conditions holds:*

1. d_2 *is even*
2. $d_1(d_2 - 9) = 4d_2$, *i.e.* $(d_1, d_2) = (5, 45), (6, 27), (7, 21), (8, 18), (10, 15), (13, 13), (16, 12), (22, 11), (40, 10)$. (*These are the examples of* [6].)

Remark 7.7. When $d_2 = 2m$, the system (2.3)–(2.6) can be realised geometrically whenever there are $m - d_1 - 1$ linearly independent vector fields on S^{m-1}. (In particular, $d_1 = m - 1$ is always possible and if $d_1 < m - 1$, then m should be even.) To see this one can simply modify the arguments on [6, pp. 240–241], replacing the vector ϵ by $m - d_1$ mutually orthogonal vectors of the form

$$\pm\epsilon_1 + \cdots \pm \epsilon_m,$$

where $\epsilon_1, \ldots, \epsilon_m$ is the standard basis of $\mathbb{R}^m$. Such a set of vectors can be obtained by evaluating the orthogonal vector fields coming from the Clifford module structure of $\mathbb{R}^m$ (over the Clifford algebra of $\mathbb{R}^{m-d_1-1}$) at the point $(1, \ldots, 1) \in S^{m-1}$.

8. Concluding remarks

We would now like to relate the series solutions of the system (2.3)–(2.6) that we have obtained to solutions of the Ricci-flat Einstein equations.

First we observe that fixing the free parameter at the top resonance ($\nu = 2$) of our Painlevé solutions is equivalent to satisfying the constraint condition $\bar{\mathsf{H}} = 0$. By Proposition 2.1 we therefore obtain local solutions of the original Ricci-flat equations.

For example, in cases (I) and II(1), the constraint equation can be written as

$$a_k + b_k + 2c_k + 2e_k = E_{11}\sum_{j=1}^{k-1} c_j c_{k-j} + 2E_{12}\sum_{j=1}^{k-1} c_j e_{k-j} + E_{22}\sum_{j=1}^{k-1} e_j e_{k-j},$$

where $\nu = k/Q = 2$. The kernel of the recursion operator $X(2)$ is spanned by (c_0, e_0, c_0, e_0). So the constraint can be satisfied provided that

$$(1, 1, 2, 2) \cdot (c_0, e_0, c_0, e_0) \neq 0.$$

But the left-hand side is

$$3(c_0 + e_0) = \frac{3}{\det E}\left(\frac{1}{d_1} + \frac{1}{d_2}\right),$$

which is clearly nonzero. The arguments for cases II(2) and II(3) are similar.

Next we analyse the local Ricci-flat metrics given by the Painlevé solutions which satisfy the constraint $\bar{\mathsf{H}} = 0$. First we consider case (I). We have

$$f_1^2 = x_1^{(1-d_2)/(n-1)} x_2^{d_2/(n-1)}, \qquad f_2^2 = x_1^{d_1/(n-1)} x_2^{(1-d_1)/(n-1)}.$$

It follows that

$$\mathrm{d}t = s^{n/(1-n)}(\text{analytic function in } s^{1/Q})\, \mathrm{d}s,$$

and hence we have $-t \sim s^{1/(1-n)}$. As s tends to 0, the arclength coordinate t tends to negative infinity. So the metric is complete at infinity. Moreover, f_i^2 are of the form t^2 (analytic function in $(-t)^{-1}$) and the volume growth is Euclidean. Indeed, our expressions for the leading terms are themselves an exact solution of (2.3)–(2.6), corresponding to the Ricci-flat cone over the product Einstein metric on $Y_1 \times Y_2$. Modulo homothety then, we have a one-parameter family of solutions when $d_1 = d_2 = 5$ and one solution in the other cases.

For case (II) we have

$$f_1^2 = x_1^{(2-d_2)/(n-1)} x_2^{(d_2-1)/(n-1)}, \qquad f_2^2 = x_1^{(d_1+1)/(n-1)} x_2^{-d_1/(n-1)}.$$

In subcase (1) we obtain the same conclusions as in case (I) with the exception that f_i^2 could be of the form t^2 (analytic function in $(-t)^{-1/k_0}$), where k_0 is a positive integer.

For subcase (2), we have

$$\mathrm{d}t = s^{d_2/(1-d_2)}(\text{analytic function in } s^{1/Q})\, \mathrm{d}s,$$

and so $-t \sim s^{1/(1-d_2)}$. It follows that as s tends to 0, the arclength t tends to minus infinity. So the metric is complete at infinity. Moreover, f_1^2 is an analytic function in $(-t)^{-1}$ while f_2^2 is of the form t^2 (analytic function in $(-t)^{-1}$). The volume growth is therefore $\sim (-t)^{d_2+1}$, as in the Euclidean Taub-NUT space. Modulo homothety we would have a one-parameter family of such solutions.

Finally for subcase (3), as s tends to 0, the arclength t tends to a finite limit t_0. Moreover, $f_1^2 \sim (t - t_0)^{-2d_2/(4d_1+d_2)}$ while $f_2^2 \sim (t - t_0)^{4d_1/(4d_1+d_2)}$.

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